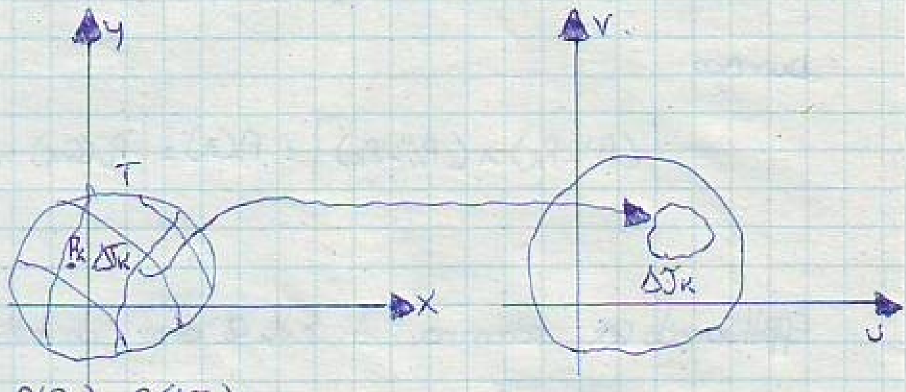
Possiamo calcolare l'espressione che cambia variabile nell'integrale doppio:

$$\int_T f(x,y)\,dT$$

con: $\begin{cases} T = \Delta T_1 \cup \dots \cup \Delta T_n \\ P_K \in \Delta T_K \end{cases}$



Quindi: $\displaystyle\int_T f(x,y)\,dT = \lim_{\delta \to 0} \sum_{1}^{n}{}_K f(P_K)\cdot A(\Delta T_K)$

Se la trasformazione è regolare, vi è una corrispondenza biunivoca tra T e J sia:

$$\Delta T_K = J(\Delta J_K) \quad , \quad \Delta J_K = T^{-1}(\Delta T_K)$$

$$\Delta T_K \subseteq T \qquad\qquad \Delta J_K \subseteq J$$

si ha che:

$$\lim_{\delta \to 0} \sum_{1}^{n}{}_K f(P_K)\cdot \int_{\Delta J_K} |P_u \wedge P_v|\,du\,dv = \sum_{1}^{n}{}_K f(P_K)\cdot A(\Delta T_K)$$

Sapendo che $P_K(u,v)$ ha tali coordinate si ha:

$$\lim_{\delta \to 0} \sum_{1}^{n}{}_K F(P_K(u,v))\,A(\Delta J_K) = \int_J F(u,v)\,du\,dv = \int_J f(x(u,v),y(u,v))\,|P_u \wedge P_v|\,du\,dv$$

Si può dunque dire:

$$\boxed{\int_T f(x,y)\,dT = \int_T f(x(u,v),y(u,v))\,\underbrace{|P_u \wedge P_v|}_{|J|}\,du\,dv}$$

Questo vale solo per le trasformazioni regolari.
Siccome la trasformazione è regolare, allora vi è una corrispondenza biunivoca tale che:

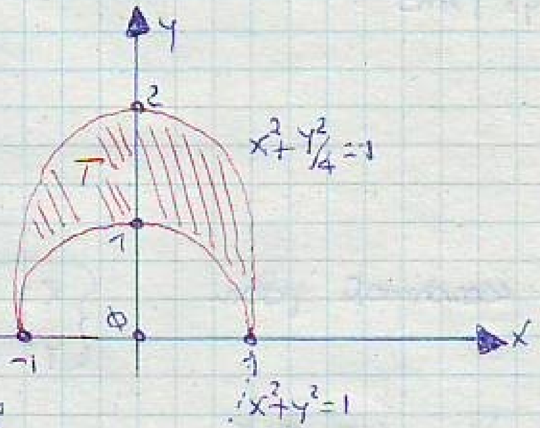
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \Rightarrow \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases} \Rightarrow (J = J^{-1})$$

* ESERCIZI:

1) Calcolare il seguente integrale doppio:

$$I=\int_T \frac{2y}{2-x}\,dT$$

dove T è il dominio indicato in figura.



1) Il dominio T è normale rispetto all'asse y. Quindi:

$$I=\int_{-1}^{1}dx\int_{\sqrt{1-x^2}}^{2\sqrt{1-x^2}}\frac{2y}{2-x}\,dy=\int_{-1}^{1}\frac{1}{2-x}\,dx\int_{\sqrt{1-x^2}}^{2\sqrt{1-x^2}}2y\,dy=\int_{-1}^{1}\frac{1}{2-x}\left[y^2\right]_{\sqrt{1-x^2}}^{2\sqrt{1-x^2}}dx=\int_{-1}^{1}\frac{1}{2-x}3(1-x^2)\,dx$$

$$=3\int_{-1}^{1}\left((x+2)+\frac{3}{x-2}\right)dx=3\left[\frac{x^2}{2}+2x+3\log(2-x)\right]_{-1}^{1}=3(4-3\log 3)$$

2) Calcolare il seguente integrale doppio:

$$I=\int_T x\,dT \qquad \text{con} \qquad T=\left\{\sqrt{1-y^2}\le x\le 2\sqrt{1-y^2}\ ;\ -1\le y\le 1\right\}$$

2) Gli estremi di integrazione esterni devono sempre essere costanti. Quindi:

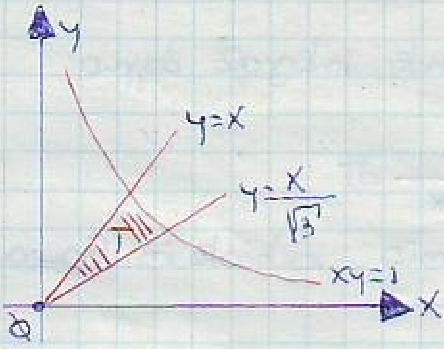
$$I=\int_{-1}^{1}dy\int_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}}x\,dx=\int_{-1}^{1}dy\left[\frac{x^2}{2}\right]_{\sqrt{1-y^2}}^{2\sqrt{1-y^2}}=\int_{-1}^{1}\left(\frac{4(1-y^2)}{2}-\frac{1-y^2}{2}\right)dy=$$

$$=\int_{-1}^{1}\left(2-2y^2-\frac{1-y^2}{2}\right)dy=\frac{1}{2}\int_{-1}^{1}\left(4-4y^2-1-y^2\right)dy=\frac{1}{2}\int_{-1}^{1}(3-3y^2)\,dy=$$

$$=\frac{1}{2}\int_{-1}^{1}3\,dy-\frac{3}{2}\int_{-1}^{1}y^2\,dy=\frac{3}{2}\left[y\right]_{-1}^{1}-\frac{1}{2}\left[y^3\right]_{-1}^{1}=\frac{3}{2}+\frac{3}{2}-\frac{1}{2}-\frac{1}{2}=\frac{4}{2}=2$$

3) Calcolare in coordinate polari il seguente integrale doppio:

$$I = \int_T y^2 \, dT$$



$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\,\text{sen}\,\vartheta \end{cases}$$

3) In coordinate polari: $\begin{cases} x = \rho\cos\vartheta \\ y = \rho\,\text{sen}\,\vartheta \end{cases}$

Quindi: $\begin{cases} \pi/6 \le \vartheta \le \pi/4 \\ 0 \le \rho \le \frac{1}{\sqrt{\text{sen}\,\vartheta\cos\vartheta}} \end{cases}$, in quanto:

$*$ $\rho\cos\vartheta = \rho\,\text{sen}\,\vartheta \Rightarrow \pi/4$

$*$ $\rho\,\text{sen}\,\vartheta = \frac{\rho\cos\vartheta}{\sqrt{3}} \Rightarrow \sqrt{3}\,\text{sen}\,\vartheta = \cos\vartheta \Rightarrow \frac{\text{sen}\,\vartheta}{\cos\vartheta} = \frac{1}{\sqrt{3}}$ $(\pi/6)$

$*$ $\rho\cos\vartheta\,\rho\,\text{sen}\,\vartheta = 1 \Rightarrow \frac{1}{\text{sen}\,\vartheta\cos\vartheta} = \rho^2 \Rightarrow \rho = \frac{1}{\sqrt{\text{sen}\,\vartheta\cos\vartheta}}$

Perciò:

$$I = \int_{\pi/6}^{\pi/4} d\vartheta \int_0^{1/\sqrt{\text{sen}\,\vartheta\cos\vartheta}} \rho^2\,\text{sen}^2\vartheta\,\rho\,d\rho = \int_{\pi/6}^{\pi/4} \text{sen}^2\vartheta\,d\vartheta \int_0^{1/\sqrt{\text{sen}\,\vartheta\cos\vartheta}} \rho^3\,d\rho = \int_{\pi/6}^{\pi/4} \text{sen}^2\vartheta\,d\vartheta \left[\frac{\rho^4}{4}\right]_0^{1/\sqrt{\text{sen}\,\vartheta\cos\vartheta}} =$$

$$= \frac{1}{4}\int_{\pi/6}^{\pi/4} \text{sen}^2\vartheta \left(\frac{1}{\sqrt{\text{sen}\,\vartheta\cos\vartheta}}\right)^4 d\vartheta = \frac{1}{4}\int_{\pi/6}^{\pi/4} \text{sen}^2\vartheta \left(\frac{1}{(\text{sen}\,\vartheta\cos\vartheta)^2}\right) d\vartheta =$$

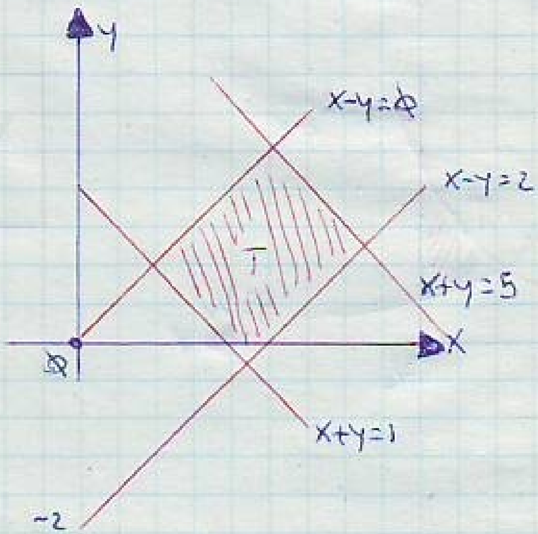
$$= \frac{1}{4}\int_{\pi/6}^{\pi/4} \cancel{\text{sen}^2\vartheta}\,\frac{1}{\cancel{\text{sen}^2\vartheta}\cos^2\vartheta}\,d\vartheta = \frac{1}{4}\int_{\pi/6}^{\pi/4} \frac{1}{\cos^2\vartheta}\,d\vartheta = \frac{1}{4}\left[\text{Tg}\,\vartheta\right]_{\pi/6}^{\pi/4} =$$

$$= \frac{\sqrt{3}-1}{4\sqrt{3}}.$$

4) Calcolare l'integrale doppio:

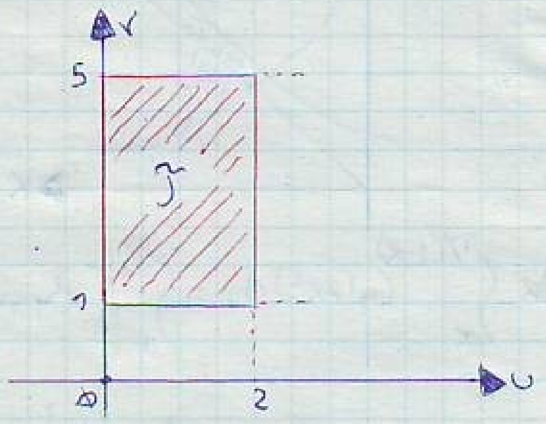
$$I = \int_T e^{x-y} \, dT$$

dove il dominio T è indicato in figura,



4) Posto: $\begin{cases} u = x-y \\ v = x+y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$

si ha:



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2} > 0$$

Quindi:

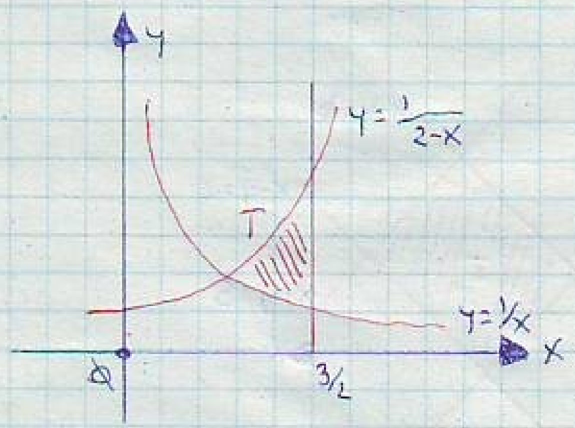
$$I = \frac{1}{2}\int_0^2 e^u du \int_1^5 dv = \frac{1}{2}\int_0^2 e^u du \left[v\right]_1^5 = \frac{1}{2}\int (5e^u - e^u) du = \frac{5}{2}(e^2-1) - \frac{1}{2}(e^2-1) =$$

$$= 2(e^2-1)$$

5) Calcolare il volume del cilindroide, con generatrici parallele all'asse z, delimitato dalla superficie:

$$z = y^2 - x^2$$

e dal dominio T tratteggiato in figura.



$$\boxed{V = \int_T |z|\, dT}$$

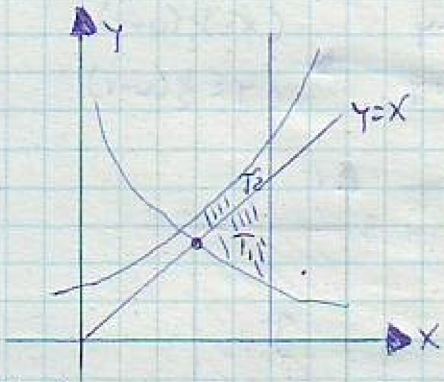
5) Studiamo il segno di z:

$z \geq 0$ in T_2

$z < 0$ in T_1

$\begin{cases} z \geq 0 & \text{per } y \geq x \\ z < 0 & \text{per } y < x \end{cases}$

$\Downarrow$



$$V = \int_{T_1} x^2 - y^2 \, dT_1 + \int_{T_2} y^2 - x^2 \, dT_2 =$$

$$= \int_1^{3/2} dx \int_{1/x}^{x} (x^2 - y^2)\, dy + \int_1^{3/2} dx \int_x^{1/(2-x)} (y^2 - x^2)\, dy = \int_1^{3/2} \left\{ \left[x^2 y - y^3/3 \right]_{1/x}^{x} + \left[y^3/3 - x^2 y \right]_x^{1/(2-x)} \right.$$

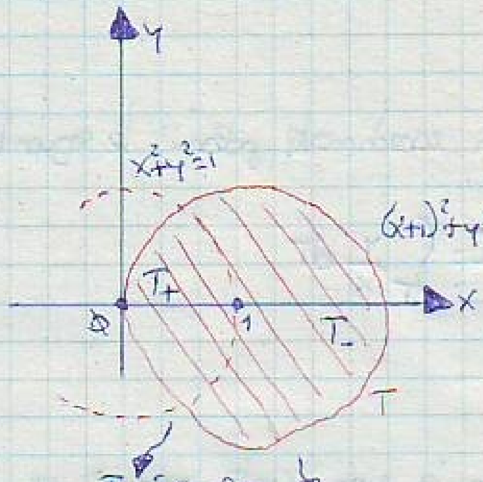
$$\left. \right\} dx = \int_1^{3/2} \left\{ x^3 - \frac{x^3}{3} - x + \frac{1}{3x^3} + \frac{1}{3(2-x)^3} - \frac{x^2}{2-x} - \frac{x^3}{3} + x^3 \right\} dx = \int_1^{3/2} \left\{ \frac{4}{3} x^3 + \right.$$

$$\left. \frac{1}{3x^3} + \frac{1}{3(2-x)^3} + \frac{2x}{x-2} \right\} dx = \left[\frac{x^4}{3} - \frac{1}{6x^2} + \frac{1}{6(2-x)^2} + 2x + 4 \log|x-2| \right]_1^{3/2} = 3 - \frac{23}{432} - 4 \log 2.$$

6) Calcolare il volume del cilindroide a generatrici parallele all'asse z, delimitato dal dominio T e dalla parte della superficie conica di equazione:

$$z = 1 - \sqrt{x^2+y^2}$$

che si proietta su T



T_+ è anche, esso simmetrico T_- è simmetrico rispetto a X.

b) $V(E) = \int_{T_+} \dots - \int_{T_-} \dots =$

$= 2\int_{T_+} \dots - 2\int_{T_-} \dots$

$$\int_{T_+} \rho(x,y)\, dT_+ = 2\int_0^{\pi/2} d\theta \int_0^{2\cos\theta} (1-\sqrt{\rho^2})\rho\, d\rho \Rightarrow \text{in quanto: } \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$* \ \rho^2 = 1 \Rightarrow \rho = 1$

$$* \ (\rho\cos\theta+1)^2 + \rho^2\sin^2\theta = 1 \Rightarrow \rho^2\cancel{\cos^2\theta} + \cancel{1} + 2\rho\cos\theta + \rho^2\cancel{\sin^2\theta} = \cancel{1} \Rightarrow \rho^2 + 2\rho\cos\theta = 0$$

$\rho = -2\cos\theta$

$$* \ \int_{T_+} \dots - \int_{T_-} \dots = \int_{T_+} \dots + \int_{T_-} \dots - 2\int_{T_-} \dots = \int_T \dots - 2\int_{T_-} \dots$$

$$\int_T \rho(x,y)\, dT = 2\int_0^{\pi/2} d\theta \int_0^{2\cos\theta} (1-\sqrt{\rho^2})\rho\, d\rho = 2\int_0^{\pi/2} \left[\frac{\rho^2}{2} - \frac{\rho^3}{3}\right]_0^{2\cos\theta} d\theta =$$

$$= 2\int_0^{\pi/2} \left(2\cos^2\theta - \frac{8}{3}\cos^3\theta\right) d\theta = 2\left[\cos\theta\sin\theta + \theta\right]_0^{\pi/2} - \frac{16}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/2} =$$

$$= \pi - \frac{16}{3} \cdot \frac{2}{3} = \pi - \frac{32}{9}$$

$$\int_{T_-} \rho(x,y)\, dT_- = 2\int_0^{\pi/3} d\theta \int_1^{2\cos\theta} (\rho - \rho^2)\, d\rho = 2\int_0^{\pi/3} d\theta \left(2\cos^2\theta - \frac{8}{3}\cos^3\theta - \frac{1}{2} + \right.$$

$$\left. + \frac{1}{3}\right) = 2\left(-\frac{1}{6}\cdot\frac{\pi}{3}\right) + 2\left[\cos\theta\sin\theta + \theta\right]_0^{\pi/3} - \frac{16}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/3} =$$

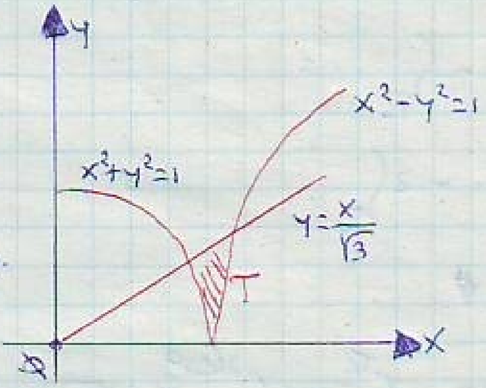
$$= \frac{\pi}{9} + 2\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right) - \frac{16}{3}\left(\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{3\cdot 8}\right) = \frac{5\pi}{9} + \frac{2\sqrt{3}}{4} - \frac{16\sqrt{3}}{6} + \frac{2\sqrt{3}}{3} = \frac{5\pi}{9} + 2\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right)$$

$$- \frac{16}{3}\left(\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{3\cdot 8}\right) = \frac{5\pi}{9} + \frac{2\sqrt{3}}{4} - \frac{16\sqrt{3}}{6} + \frac{2\sqrt{3}}{3} = \frac{5\pi}{9} + \frac{\sqrt{3}}{2}\left[1 - \frac{16}{3} + \frac{4}{3}\right] = \frac{5\pi}{9} - \frac{3\sqrt{3}}{2}$$

$$V(E) = \pi - \frac{32}{9} - 2\left[\frac{5\pi}{9} - \frac{3\sqrt{3}}{2}\right] = -\frac{\pi}{9} - \frac{32}{9} + 3\sqrt{3}.$$

7) Calcolare in coordinate polari il seguente integrale doppio:

$$I=\int_T \frac{y}{x}\,dT$$



e indicarne il significato geometrico.

7) $\begin{cases} x=\rho\cos\theta \\ y=\rho\,\mathrm{sen}\,\theta \end{cases} \Rightarrow$ $*$ $\rho^2=1 \Rightarrow \rho=1 \quad (\rho^2\cos^2\theta+\rho^2\mathrm{sen}^2\theta=1)$

$*$ $\rho^2\cos^2\theta-\rho^2\mathrm{sen}^2\theta=1 \Rightarrow \rho^2(\cos^2\theta-\mathrm{sen}^2\theta)=1$

$\Downarrow$

$$\rho=\frac{1}{\sqrt{(\cos^2\theta-\mathrm{sen}^2\theta)}}=\frac{1}{\sqrt{\cos 2\theta}}$$

$*$ $\rho\,\mathrm{sen}\,\theta=\frac{\rho\cos\theta}{\sqrt{3}} \Rightarrow \theta=\pi/6$

$\Downarrow$

$$I=\int_0^{\pi/6}d\theta\int_1^{1/\sqrt{\cos 2\theta}}\frac{\rho\,\mathrm{sen}\,\theta}{\rho\cos\theta}\,\rho\,d\rho=\int_0^{\pi/6}\frac{\mathrm{sen}\,\theta}{\cos\theta}\,d\theta\left[\frac{\rho^2}{2}\right]_1^{1/\sqrt{\cos 2\theta}}=\frac{1}{2}\int_0^{\pi/6}\frac{\mathrm{sen}\,\theta}{\cos\theta}\left(\frac{1}{\cos 2\theta}-\frac{1}{2}\right)d\theta=$$

$$=\frac{1}{2}\int_0^{\pi/6}\frac{\mathrm{sen}\,\theta\,d\theta}{\cos\theta(2\cos^2\theta-1)}+\frac{1}{2}\Big[\log|\cos\theta|\Big]_0^{\pi/6};$$

Posto: $\cos\theta=t \Rightarrow dt=-\mathrm{sen}\,\theta\,d\theta \Rightarrow -\frac{1}{2}\int_1^{\sqrt{3}/2}\frac{dt}{t(2t^2-1)}+\frac{1}{2}\log\frac{\sqrt{3}}{2}$.

Siccome:

$$\frac{1}{t(2t^2-1)}=\frac{1}{2}\left(\frac{-2}{t}+\frac{1}{t-1/\sqrt{2}}+\frac{1}{t+1/\sqrt{2}}\right)$$

$\Downarrow$

$$I=-\frac{1}{4}\left[-2\log t+\log\left(t^2-\frac{1}{2}\right)\right]_1^{\sqrt{3}/2}+\frac{1}{2}\log\frac{\sqrt{3}}{2}=\frac{1}{2}\log 3-\frac{3}{4}\log 2.$$

Siccome:

$f(x,y)=y/x\geq 0$ in T, il risultato trovato rappresenta il volume del cilindroide

$$E=\{(x,y,z):(x,y)\in T \text{ e } 0\leq z\leq y/x\}$$